Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2015 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

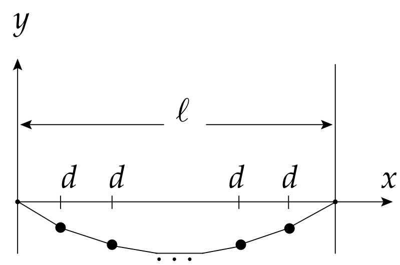
Each problem counts for 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages.
Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Wave propagation in an (an)harmonically coupled system of N masses



Consider a massless string of length ℓ_0 with string constant κ, containing N masses, each with mass m. The string is stretched to a length $\ell > \ell_0$ and attached to two opposite walls, as in the figure. In equilibrium, the distance from each wall to the nearest mass is d, as is the distance between two neighbouring masses. The position of the masses is (x_k, y_k) with $k = 1, \cdots, N$, and the endpoints are fixed at $(x_0, y_0) = (0, 0)$ and $(x_{N+1}, y_{N+1}) = (\ell, 0)$. Neglect gravity. The potential is $V(\xi_k, y_k) = \frac{1}{2}\kappa \sum_{k=1}^{N+1} (\ell_k - a)^2$, where $a = \frac{\ell_0}{N+1}$ and ℓ_k is the length of the segment of the string between the k^{th} mass and the $(k-1)^{\text{th}}$ mass, and $\xi_k = x_k - kd$.

a) (2 points) Write down the Lagrangian L without any approximations. Then expand to **third** order in small deviations (ξ_k, y_k).

b) (6 points) Consider first harmonic motion (due to the terms in L which are quadratic in ξ_k and y_k). Write down the equation of motion for ξ_j and y_j. Expand x_j and y_j into normal modes. What are the integration constants? (Hint: Try the ansatz ξ_j proportional to $e^{ij\alpha}$, and similarly for y_j.)

c) (6 points) What are the frequencies ω as a function of j? Take the continuum limit $N \to \infty$, keeping the total mass $M = Nm$ fixed. Show that one obtains the wave equation. What are the dispersion relations for $\omega(k)$ as a function of the wave number k for transverse and longitudinal modes?

d) (6 points) Now consider the cubic anharmonic term in L. Obtain the wave equation for longitudinal waves and transverse waves with the contribution from this term. Describe the difference in the propagation of a longitudinal pulse and the propagation of a transverse pulse due to the anharmonic term.

Solution

a) The Lagrangian is

$$L = \frac{1}{2} m \sum_{k=1}^{N} \left(\dot{\xi}_k^2 + \dot{y}_k^2 \right) - \frac{1}{2} \kappa \sum_{k=1}^{N+1} \left(\sqrt{(d + \xi_k - \xi_{k-1})^2 + (y_k - y_{k-1})^2} - a \right)^2 . \tag{1}$$

Expanding the Lagrangian to third order we get

$$L = \frac{1}{2} m \sum_{k=1}^{N} \left(\dot{\xi}_k^2 + \dot{y}_k^2 \right) - \frac{1}{2} \kappa \sum_{k=1}^{N+1} \left[(\Delta \xi_k)^2 + \frac{d-a}{d} (\Delta y_k)^2 + \frac{a}{d^2} \Delta \xi_k (\Delta y_k)^2 \right] \tag{2}$$

where $a = \frac{\ell_0}{N+1}$, $\Delta \xi_k = \xi_k - \xi_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$. (As a check, note that for $a = 0$ all anharmonic terms should vanish, because in that case $V = \frac{1}{2} \kappa \sum_{k=1}^{N+1} \ell_k^2$.)

b) The Lagrangian equations of motion are

$$\left. \begin{aligned} m \ddot{\xi}_j &= \kappa \left(\xi_{j+1} - 2\xi_j + \xi_{j-1} \right) \\ m \ddot{y}_j &= \kappa \left(\tfrac{d-a}{d} \right) \left(y_{j+1} - 2 y_j + y_{j-1} \right) \end{aligned} \right\} \text{ Set } \left\{ \begin{aligned} \xi_j &= \left(\mathrm{Re}\, e^{ij\alpha - i\delta} \right) \cos(\omega t + a) \\ y_j &= \left(\mathrm{Re}\, e^{ij\beta - i\delta'} \right) \cos(\Omega t + b) \end{aligned} \right. \tag{3}$$

Then

$$\begin{aligned} -m\omega^2 &= \kappa \left(e^{i\alpha} - 2 + e^{-i\alpha} \right) \Rightarrow \omega^2 = \tfrac{4\kappa}{m} \sin^2 \tfrac{\alpha}{2} \\ -m\Omega^2 &= \kappa \left(\tfrac{d-a}{d} \right) \left(e^{i\beta} - 2 + e^{-i\beta} \right) \Rightarrow \Omega^2 = \tfrac{4\kappa}{m} \left(\tfrac{d-a}{d} \right) \sin^2 \tfrac{\beta}{2} . \end{aligned} \tag{4}$$

The boundary condition give $\delta = \delta' = \frac{\pi}{2}$ and $\sin(N+1)\alpha = \sin(N+1)\beta = 0$, hence

$$\alpha = \beta = \frac{k\pi}{N+1} \text{ for } k = 1, 2, \cdots, N.$$

So the normal modes with longitudinal frequencies $\omega_L^{(k)}$ are given by

$$\begin{aligned} x_j &= jd + \sum_{k=1}^{N} \left(\sin \frac{j\pi k}{N+1} \right) A_k \cos \left(\omega_L^{(k)} t + a_k \right) \\ \left(\omega_L^{(k)} \right)^2 &= \tfrac{4\kappa}{m} \sin^2 \tfrac{\pi k}{2(N+1)} . \end{aligned} \tag{5}$$

For the transversal modes with frequencies $\omega_T^{(k)}$ one gets

$$\begin{aligned} y_j &= \sum_{k=1}^{N} \left(\sin \frac{j\pi k}{N+1} \right) B_k \cos \left(\omega_T^{(k)} t + b_k \right) \\ \left(\omega_T^{(k)} \right)^2 &= \tfrac{4\kappa}{m} \left(\tfrac{d-a}{d} \right) \sin^2 \tfrac{\pi k}{2(N+1)} \,. \end{aligned} \tag{6}$$

There are $2N$ normal modes and $4N$ integration constants: a_k, A_k, b_k and B_k. The frequencies of the longitudinal modes are larger than the corresponding frequencies of the transversal modes if $\ell_0 < \ell$.

c) Setting $\xi_j(t) = \xi(x,t)$ and $y_j(t) = y(x,t)$ with $x = jd$, and $\xi_{j+1} - \xi_j = d \frac{\partial}{\partial x} \xi(x,t)$ at $x = jd + \frac{1}{2}d$, one obtains

$$\xi_{j+1} - 2\xi_j + \xi_{j-1} = d^2 \tfrac{\partial^2}{\partial x^2} \xi(x,t) \,.$$

Then the equations of motion become

$$\begin{aligned} m \tfrac{\partial^2}{\partial t^2} \xi(x,t) &= \kappa d^2 \tfrac{\partial^2}{\partial x^2} \xi(x,t) \\ m \tfrac{\partial^2}{\partial t^2} y(x,t) &= \kappa d^2 \left(\tfrac{d-a}{d} \right) \tfrac{\partial^2}{\partial x^2} y(x,t) \,. \end{aligned} \tag{7}$$

Setting $\frac{m}{d} = \rho =$ mass density, and $\frac{\kappa d}{\ell} = f =$ string tension per unit length, we get

$$\begin{aligned} \left(\rho \tfrac{\partial^2}{\partial t^2} - f\ell \tfrac{\partial^2}{\partial x^2} \right) \xi(x,t) &= 0 \\ \left(\rho \tfrac{\partial^2}{\partial t^2} - f\left(\ell - \ell_0\right) \tfrac{\partial^2}{\partial x^2} \right) y(x,t) &= 0 \,. \end{aligned} \tag{8}$$

Hence the wave velocities are

$$v_L^2 = \frac{f\ell}{\rho} \,; \qquad v_T^2 = \frac{f(\ell - \ell_0)}{\rho} \,. \tag{9}$$

Defining a wave vector $\hat{k}$ by

$$\xi_j^{(k)} \sim e^{ij\left(\frac{\pi k}{N+1}\right)} = e^{i(jd)\frac{\pi k}{(N+1)d}} = e^{ix\frac{\pi k}{\ell}} = e^{ix\hat{k}}$$

we identify $\hat{k} = \frac{\pi k}{\ell}$. Then

$$\left(\omega_L^{(k)} \right)^2 = \omega_L^2(\hat{k}) = \frac{4\kappa}{m} \sin^2 \frac{\pi k}{2(N+1)} = 4\frac{f\ell}{d}\frac{1}{\rho d} \sin^2 \frac{\hat{k}\ell}{2(N+1)} = 4\frac{f\ell}{\rho d^2} \sin^2 \frac{\hat{k}d}{2} \simeq \frac{f\ell}{\rho} \hat{k}^2 \tag{10}$$

So there is linear dispersion, $\omega_L = v_L \hat{k}$. Similarly $\omega_T = v_T \hat{k}$.

d) The equation of motion for ξ_j becomes

$$m\ddot{\xi}_j = \kappa\left(\xi_{j+1} - 2\xi_j + \xi_{j-1}\right) + \tfrac{a\kappa}{2d^2}\left[\left(\Delta y_{j+1}\right)^2 - \left(\Delta y_j\right)^2\right] \tag{11}$$

In the continuum limit one obtains

$$\left(\rho\frac{\partial^2}{\partial t^2} - f\ell\frac{\partial^2}{\partial x^2}\right)\xi(x,t) = \frac{a\kappa}{2d^3}\left(\Delta y_{j+1} - \Delta y_j\right)\left(\Delta y_{j+1} + \Delta y_j\right). \tag{12}$$

With $\Delta y_{j+1} - \Delta y_j = y_{j+1} - 2y_j + y_{j-1} = d^2\frac{\partial^2 y}{\partial x^2}$ and $\Delta y_{j+1} + \Delta y_j = y_{j+1} - y_{j-1} \simeq 2d\frac{\partial y}{\partial x}$, one obtains

$$\begin{aligned}\left(\rho\frac{\partial^2}{\partial t^2} - f\ell\tfrac{\partial^2}{\partial x^2}\right)\xi(x,t) &= a\kappa\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial^2 y}{\partial x^2}\right)\\ &= \left(\frac{\ell_0}{N+1}\right)\left((N+1)f\right)\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\ell_0 f}{2}\frac{\partial}{\partial x}\left(\left(\frac{\partial y}{\partial x}\right)^2\right).\end{aligned} \tag{13}$$

An interesting effect now occurs: if at time $t = 0$ one has a transverse pulse $y(x,t)$ moving to the right with velocity v_T, at later times a longitudinal wave $\xi(x,t)$ is created. It is given by

$$\xi(x,t) = \int G(x-x', t-t')\left[-\frac{1}{2}\left(\frac{\ell_0}{\ell}\right)\frac{\partial}{\partial x'}\left(\frac{\partial y(x',t')}{\partial x'}\right)^2\right]dx'dt'$$

where $G(x-x', t-t')$ is the retarded Green function

$$\begin{aligned}G(x-x', t-t') &= \frac{1}{\partial_x^2 - v_L^{-2}\partial_t^2}\delta(x-x')\delta(t-t')\\ &= -\tfrac{1}{4}v_L\left[\epsilon\big(x-x'+v_L(t-t')\big) - \epsilon\big(x-x'-v_L(t-t')\big)\right]\theta(t-t')\end{aligned}$$

The pulse moves slower than the speed v_L in the Green function, so the integral over x' and t' is finite. However, if at $t = 0$ there is only a longitudinal pulse, the anharmonic term in the wave equation of transverse waves vanishes, so no transverse wave will be generated.

Classical Mechanics 2

Point particle moving on a sphere

Consider a point particle with mass m, moving on a fixed sphere with radius R in the gravitational field of the Earth.

a) (3 points) Write down the Lagrangian in spherical coordinates (r, θ, ϕ). Write down explicit expressions for the two conserved quantities.

b) (6 points) Using the expressions in a), write the time t as a function of θ, and ϕ as a function of θ. You will find integrals which can not be evaluated in closed form. That is OK, just write them down.

c) (4 points) If the particle moves on a horizontal circle with azimuthal angle θ, what is the angular velocity of this periodic orbit?

d) (7 points) If the particle has a z-component M_z of the orbital angular momentum, all its trajectories will lie between two horizontal circles on the sphere. Find an equation for the position of these circles. How many solutions does this equation have, and what can you say on physical grounds about these solutions?

Solution

a) $L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgR\cos\theta.$

$$M_z = m(x\dot{y} - y\dot{x}) = mR^2\sin^2\theta\dot{\phi}$$
$$E = \tfrac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgR\cos\theta\,.$$

b)

$$\dot{\theta}^2 = \frac{E - mgR\cos\theta}{\frac{1}{2}mR^2} - \frac{M_z^2}{m^2R^4\sin^2\theta} = \frac{1}{\frac{1}{2}mR^2}\left(E - \mathcal{U}_{eff}(\theta)\right)$$

where

$$\mathcal{U}_{eff}(\theta) = \frac{M_z^2}{2mR^2\sin^2\theta} + mgR\cos\theta\,.$$

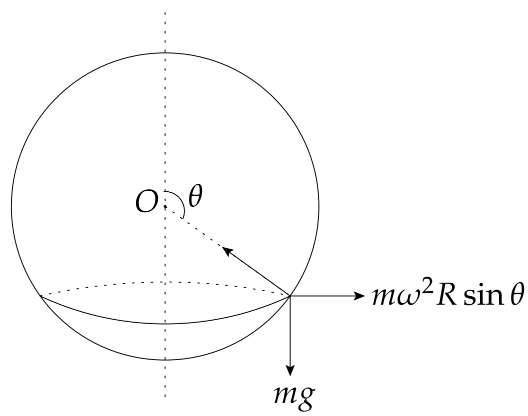
Hence

$$t = \sqrt{\tfrac{1}{2}mR^2} \int \frac{d\theta}{\sqrt{E - \mathcal{U}_{eff}(\theta)}} \, .$$

From $\frac{\dot{\phi}}{\dot{\theta}} = \frac{d\phi}{d\theta}$ and the expression for $\dot{\phi}$ and $\dot{\theta}$ we find an equation for ϕ as a function of θ

$$\phi = \int \frac{M_z}{mR^2 \sin^2\theta} \frac{\sqrt{\frac{1}{2}mR^2}}{\sqrt{E - \mathcal{U}_{eff}(\theta)}} d\theta = \frac{M_z}{R\sqrt{2m}} \int \frac{d\theta}{\sin^2\theta \sqrt{E - \mathcal{U}_{eff}(\theta)}} \, .$$

c) The reaction force $\vec{F}$ is in the direction of the center O, and must balance the gravitational force and the centrifugal force on the particle.



$$\begin{aligned} F|\cos\theta| &= mg \\ F\sin\theta &= m\omega^2 R \sin\theta \, . \end{aligned}$$

Hence

$$\omega^2 = \frac{F}{mR} = \frac{mg}{mR} \frac{1}{|\cos\theta|} = \frac{g}{R} \frac{1}{|\cos\theta|} \, .$$

d) Since $\dot{\theta} = \sqrt{\frac{2}{mR^2}\big(E - \mathcal{U}_{eff}(\theta)\big)}$, the orbits of the point particle touch the circles when $E = \mathcal{U}_{eff}(\theta)$. This happens when

$$\frac{M_z^2}{2mR^2} + mgR\cos\theta(1 - \cos^2\theta) = E(1 - \cos^2\theta) \, .$$

This is a cubic equation for $\cos\theta$, which has 3 roots. On physical grounds there should be two roots which are real and lie between -1 and $+1$, corresponding to the upper circle on the sphere (where $\cos\theta$ can be positive, zero or negative) and the lower circle (where $\cos\theta < 0$). The third root must then also be real, but it can not be a turning point for the motion of the point particle, so it should be a real root, which lies outside $(-1, +1)$.

To work this out, introduce dimensionless variables $\frac{E}{mgR} = \epsilon$ and $\frac{M_z^2}{2gm^2R^3} = c$, and denote $\cos\theta$ by x. Then the equation becomes

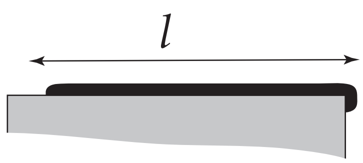
$$x^3 - \epsilon x^2 - x + (\epsilon - c) = 0\,.$$

If at $t = 0$ the particle bounces off the equator with energy $\epsilon = c$, one expects that one circle is the equator $x_1 = 0$, and the other circle is given by $-1 < x_2 < 0$. The three solutions of $x^3 - \epsilon x^2 - x = 0$ are $x = 0$ and $x = \frac{\epsilon}{2} \mp \sqrt{1 + \left(\frac{\epsilon}{2}\right)^2}$, and the first two roots indeed yield these two circles. The third root x_3 is indeed real, and $x_3 > 1$.

Classical Mechanics 3

Rope sliding off a table

An ideal (flexible, uniform, frictionless, etc.) rope of the length l and mass M starts sliding off an ideal frictionless table as shown in the figure (the rope is initially at rest, the gravitational acceleration is g, the size of the piece of the rope initially hanging off the table is y_0).



a) (2 points) Introduce some generalized coordinate and write down the Lagrangian of the system.

b) (2 points) Derive the Euler-Lagrange equations of motion.

c) (2 points) Write down the energy of the system.

d) (6 points) Calculate the time T for the rope to slide half way off the table.

e) (8 points) Compute the horizontal component of the reaction force of the table at the moment when the rope is exactly half way off the table. *Hint:* calculate the rate of the change of the momentum of the rope.

Solution

a) Let y be the amount of rope hanging over the edge of the table,which will then have mass, $m_y = \frac{yM}{l}$. Then the potential energy of the rope is $U = -\frac{1}{2}m_y g y = -\frac{Mgy^2}{2l}$ and the Lagrangian:

$$L = \frac{1}{2}M\dot{y}^2 + \frac{1}{2l}Mgy^2$$

b) Using standard formulas we obtain

$$\ddot{y} = \omega^2 y\,,$$

where $\omega = \sqrt{g/l}$.

c) The energy is given by the sum of the kinetic and potential energies

$$E = \frac{1}{2}M\dot{y}^2 - \frac{1}{2l}Mgy^2 \,.$$

d) Since $\frac{\partial L}{\partial t} = 0$, E is conserved. Thus $E = -\frac{Mgy_0^2}{2l} = -\frac{Mgy^2}{2l} + \frac{1}{2}M\dot{y}^2$. This gives a first integral of the motion,

$$\dot{y}^2 = \omega^2(y^2 - y_0^2) \,.$$

Then integrating $\int dt = \int (dy/\dot{y})$ we get

$$T = \omega^{-1}\int_{y_0}^{l/2} \frac{dy}{\sqrt{y^2 - y_0^2}} = \sqrt{\frac{l}{g}} \ln\left(\sqrt{\frac{l}{2y_0}} + \sqrt{\frac{l}{2y_0} - 1}\right) .$$

However, the integrals are more elementary if we use $\ddot{y} = \omega^2 y$ to write

$$y(t) = Ae^{\omega t} + Be^{-\omega t}.$$

The initial conditions $\dot{y}(t=0)$ and $y(0) = y_0$ tell us that $A = B = y_0/2$, so

$$y(t) = y_0 \cosh(\omega t) \text{ and } \omega t = \cosh^{-1}\frac{y}{y_0} \text{ and } T = \sqrt{\frac{l}{g}}\cosh^{-1}\frac{l}{2y_0}.$$

This is in fact the same as the more complicated looking natural log found by integration of the conserved energy.

e) The horizontal component p_x of the momentum of the rope is changing (increasing, initially) in time, and this $\dot{p}_x$ is the "reaction force" F_x that the table exerts on the rope. The horizontal momentum consists of the mass of that fraction of the rope which moves horizontally on the table, times the horizontal velocity which equals the vertical velocity $\dot{y}$,

$$p_x = M\frac{l-y}{l}\dot{y} \,.$$

Taking the time derivative of the momentum and using equation of motion and energy conservation, we have

$$\begin{aligned}
\frac{dp_x}{dt} &= \frac{M}{l}\left[(l-y)\ddot{y} - \dot{y}^2\right] \\
&= \frac{M}{l}\left[(l-y)\omega^2 y - \omega^2(y^2 - y_0^2)\right] \\
&= Mg\frac{1}{l^2}\left[y(l-y) - (y^2 - y_0^2)\right] \,.
\end{aligned}$$

This is the horizontal component of the reaction force. Taking it at $y = l/2$ we obtain

$$F_x = Mg\frac{y_0^2}{l^2}\,.$$

It is interesting to note that at the point $y(t)$ where $\frac{dp_x}{dt} = 0$, that is at right after $y = l/2$ if y_0 is small, the horizontal reaction force will vanish. This means that the rope will depart from the table and after that the equations we derived are invalid.

Electromagnetism 1

Torque on a cylinder

The constitutive relation is a relation between the macroscopic electrical current density in a medium and the applied fields. Recall that for a normal isotropic conductor *at rest* in an electric ($\boldsymbol{E}$) and magnetic field ($\boldsymbol{B}$) the constitutive relation in a linear response approximation is known as Ohm's law: $\boldsymbol{J} = \sigma \boldsymbol{E}$.

a) (2 points) For most materials, a symmetry principle forbids a generalized Ohm's law in the rest frame of the material of the form:

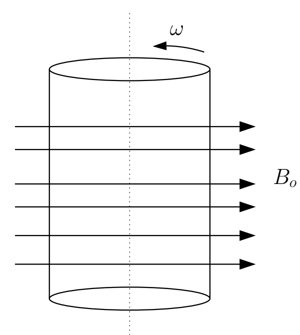
$$\boldsymbol{J} = \sigma \boldsymbol{E} + \sigma_B \boldsymbol{B}\,. \tag{1}$$

Explain.

b) (6 points) By making a Lorentz transformation for small velocities, deduce the familiar constitutive relation for a normal conductor *moving* non-relativistically with velocity $\boldsymbol{u}$ in an electric and magnetic field from the rest frame constitutive relation, Eq. ().

c) (4 points) Now consider a solid conducting cylinder of radius R and conductivity σ rotating rather slowly with constant angular velocity ω in a uniform magnetic field B_o perpendicular to the axis of the cylinder as shown below. Determine the current flowing in the cylinder.

d) (8 points) Determine the torque required to maintain the cylinder's constant angular velocity. Assume that the skin depth is much larger than the radius of the cylinder.



Solution

a) Parity forbids a constitutive relation including a magnetic field. Specifically, σ_B would have to be a pseudo-scalar, since $\boldsymbol{J}$ is a vector and $\boldsymbol{B}$ is a pseudo-vector. But, if the interactions of the medium are invariant under parity, and the ground state is parity symmetric, then medium expectation value of any pseudoscalar quantity is zero.

b) In a frame where the conductor is at rest

$$\underline{\boldsymbol{J}} = \sigma \underline{\boldsymbol{E}} \tag{2}$$

the charge density $\rho = 0$. Make a Lorentz transformation from the conductor's rest frame to the lab frame, *i.e.* a frame moving with velocity $-\boldsymbol{u}$ relative to the conductor, so that the lab observer sees the conductor moving with velcoity $\boldsymbol{u}$. We have

$$J^\mu = \Lambda^\mu_{\ \nu} \underline{J}^\nu \,. \tag{3}$$

Here the $\underline{J}$ are the currents in the conductor frame, J are the currents in the lab frame.

To first order in $\boldsymbol{u}$ the Lorentz transformation matrix is

$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & \gamma u \\ \gamma u & \gamma \end{pmatrix} \approx \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix} \tag{4}$$

Thus

$$\boldsymbol{J} \approx \boldsymbol{u} \underbrace{\underline{\rho}}_{=0} + \underline{\boldsymbol{J}} = \sigma \underline{\boldsymbol{E}} \tag{5}$$

We need to use the Lorentz transformation rule to relate $\underline{\boldsymbol{E}}$ to $\boldsymbol{E}$ and $\boldsymbol{B}$.

The transformation rules for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are

$$E_\parallel = \underline{E}_\parallel \tag{6}$$

$$B_\parallel = \underline{B}_\parallel \tag{7}$$

$$E_\perp = \gamma \underline{E}_\perp - \gamma \boldsymbol{u}/c \times \underline{\boldsymbol{B}} \tag{8}$$

$$B_\perp = \gamma \underline{B}_\perp + \gamma \boldsymbol{u}/c \times \underline{\boldsymbol{E}} \tag{9}$$

and the inverse results

$$\underline{E}_{\parallel} = E_{\parallel} \tag{10}$$

$$\underline{B}_{\parallel} = B_{\parallel} \tag{11}$$

$$\underline{E}_{\perp} = \gamma E_{\perp} + \gamma \boldsymbol{u}/c \times \boldsymbol{B} \approx E_{\perp} + \boldsymbol{u}/c \times \boldsymbol{B} \tag{12}$$

$$\underline{B}_{\perp} = \gamma B_{\perp} - \gamma \boldsymbol{u}/c \times \boldsymbol{E} \tag{13}$$

So the constitutive relation becomes to first order

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \frac{\boldsymbol{u}}{c} \times \boldsymbol{B}) \tag{14}$$

Clearly the constitutive relation takes the form $\boldsymbol{J} = \sigma \boldsymbol{f}$ where $\boldsymbol{f}$ is the Lorentz force.

c) Using the result

$$\boldsymbol{J} = \sigma(\boldsymbol{u}/c \times \boldsymbol{B}_o) \,, \tag{15}$$

we find in cylindrical coordinates

$$\boldsymbol{J}(\rho, \phi) = -\sigma \frac{\omega \rho B_o}{c} \cos\phi \hat{\boldsymbol{z}} \,. \tag{16}$$

We see that the electrons (which carry negative charge) flow up the cylinder at $\phi = 0$ and down the cylinder at $\phi = \pi$.

d) The Lorentz force on the current induces a torque:

$$\tau = \int d^3\boldsymbol{r}\, \boldsymbol{r} \times \left(\frac{\boldsymbol{J}}{c} \times \boldsymbol{B}_o\right) \,, \tag{17}$$

$$= L \int \rho d\rho d\phi \left[\frac{\boldsymbol{J}}{c}(\boldsymbol{r} \cdot \boldsymbol{B}_o) - (\boldsymbol{r} \cdot \boldsymbol{J}/c)\boldsymbol{B}_o\right] \,, \tag{18}$$

where L is the length of the cylinder. The second term in square braces integrates to zero while the first terms gives

$$\tau = L \int_0^R \rho d\rho \int d\phi \left[(-\sigma \frac{\omega \rho B_o}{c^2} \cos\phi \hat{\boldsymbol{z}})(\rho \cos\phi B_o)\right] \,, \tag{19}$$

$$= -L\hat{\boldsymbol{z}} \frac{(\pi \sigma \omega R^4 B_o^2)}{4c^2} \,. \tag{20}$$

This is the torque by the magnetic field on the cylinder. To maintain a constant angular velocity we need an external torque per unit length of

$$\frac{\tau}{L} = +\hat{\boldsymbol{z}} \frac{(\pi \sigma \omega R^4 B_o^2)}{4c^2} \,. \tag{21}$$

Notes:

- An alternative way to derive this is to equate the work done per time by the external torque, $\tau \cdot \omega$, with the energy dissipation
$$\tau \cdot \omega = \int d^3\boldsymbol{r}\frac{\boldsymbol{J}\cdot\boldsymbol{J}}{\sigma} \tag{22}$$
$$= L\frac{\sigma\omega^2}{4c^2}B_o^2\pi R^4 \tag{23}$$

- We next evaluate this numerically for copper. Expressing the torque in terms of the skin depth (which is taken from Wikipedia):
$$\delta = \sqrt{\frac{2c^2}{\sigma\omega}} = 6.5\,\text{cm}/\sqrt{f_{Hz}} \tag{24}$$
We find
$$\frac{\tau}{L} = \frac{R^4 B_o^2}{\delta^2}\frac{\pi}{2} \tag{25}$$
Converting to MKS and Tesla
$$B_o^2 \to \frac{B_o^2}{\mu_o} = 1\,\frac{J}{m^3}\,8\times 10^5\left(\frac{B_o}{\text{Tesla}}\right)^2 \tag{26}$$
So we find
$$\frac{\tau}{L} \approx 3\,\text{N}\left(\frac{R}{\text{cm}}\right)^4\left(\frac{f}{Hz}\right)\left(\frac{B_o}{\text{Tesla}}\right)^2 \qquad \text{with} \qquad R \ll \frac{6.5\,\text{cm}}{\sqrt{f \text{ in Hz}}} \tag{27}$$

It is also interesting to calculate the current flowing through each hemi-cylinder of the wire.
$$\frac{I}{c} = \int \rho d\rho \int_{-\pi/2}^{\pi/2} d\phi \boldsymbol{J}(\rho,\phi)/c \tag{28}$$
$$= -\frac{2}{3}\sigma\frac{\omega R^3 B_o}{c^2}\hat{\boldsymbol{z}} \tag{29}$$
$$= -\frac{4}{3}\frac{R^3 B_o}{\delta^2} \tag{30}$$
Or in MKS
$$\frac{I}{c} \to \sqrt{\mu_o} I \tag{31}$$
$$\boldsymbol{B} \to \frac{\boldsymbol{B}}{\sqrt{\mu_o}} \tag{32}$$

which evaluates to a shockingly large current

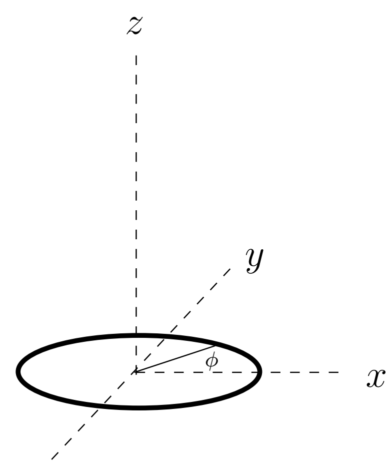
$$I = -\frac{4}{3}\frac{R^3 B_o}{\delta^2 \mu_o} \tag{33}$$

$$= 310\text{Amps}\left(\frac{f}{\text{Hz}}\right)\left(\frac{B}{\text{Tesla}}\right)\left(\frac{R}{\text{cm}}\right)^3 \tag{34}$$

Electromagnetism 2

Oscillating current on a ring

A current is driven through a ring of radius R in the xy plane (see below). Using complex notation, the current has a harmonic time dependence, $\boldsymbol{J}(t, \boldsymbol{r}) = e^{-i\omega t}\boldsymbol{J}(\boldsymbol{r})$, and the spatial dependence is $\boldsymbol{J}(\boldsymbol{r}) = I_0 \sin(\phi)\, \delta(\rho - R)\delta(z)\hat{\boldsymbol{\phi}}$.



a) (4 points) Sketch the current flow at time $t = 0$ and $t = \pi/\omega$, and determine the charge density $\rho(t, \boldsymbol{r})$. Show that it corresponds to an oscillating electric dipole, and determine the electric dipole moment.

b) In the long wavelength limit, and in the radiation zone, determine each of the following quantities in the xz plane at $y = 0$:

   (a) (6 points) The vector potential $\boldsymbol{A}(t, \boldsymbol{r})$ in the Lorentz gauge.

   (b) (4 points) The magnetic field $\boldsymbol{B}(t, \boldsymbol{r})$.

   (c) (4 points) The (time averaged) angular distribution of the radiated power, $dP/d\Omega$.

c) (2 points) What is the polarization of the radiated electric field when viewed along the z axis ?

Solution

We use Heavyside-Lorentz units.

a) Using current conservation, $\partial_t \rho + \nabla \cdot \boldsymbol{J} = 0$ and a harmonic time dependence, $\rho(t, \boldsymbol{r}) = e^{-i\omega t}\rho(\boldsymbol{r})$,

$$- i\omega\rho(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = -\frac{1}{R}\frac{\partial}{\partial\phi} J^{\phi} \,. \tag{1}$$

Thus

$$\rho(\boldsymbol{r}) = -\frac{I_o \cos\phi}{-i\omega R}\delta(z)\delta(\rho - R) \tag{2}$$

Note, the charge distribution gives rise to a net dipole moment

$$\boldsymbol{p} = \int d^3\boldsymbol{r}\rho(\boldsymbol{r})\boldsymbol{r} = \frac{I_o R}{-i\omega}(-\pi\hat{\boldsymbol{x}}) \tag{3}$$

pointed along the negative $\hat{\boldsymbol{x}}$ direction. If this is recognized then the remainder of this problem is just quoting the results of the electric dipole radiation.

b) a) In the dipole approximation we have

$$\begin{aligned} \boldsymbol{A}(t, \boldsymbol{r}) &= \frac{e^{-i\omega t + ikr}}{4\pi r} \int d^3\boldsymbol{r}' \boldsymbol{J}(\boldsymbol{r}')/c \\ &= \frac{e^{-i\omega t + ikr}}{4\pi r} \int \rho d\rho d\phi dz \, \hat{\boldsymbol{\phi}} \, (I_0/c) \sin\phi \, \delta(\rho - R)\delta(z) \,. \end{aligned} \tag{4}$$

With $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$ we obtain

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e^{-i\omega t + ikr}}{4\pi r} R(I_0/c)\pi \, (-\hat{\boldsymbol{x}}) \,, \tag{5}$$

$$= \frac{e^{-i\omega t + ikr}}{4\pi r} \frac{-i\omega}{c} \boldsymbol{p} \tag{6}$$

b) Then

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \,, \tag{7}$$

$$= \boldsymbol{n} \times \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(t, \boldsymbol{r}) \,, \tag{8}$$

$$= \frac{e^{-i\omega t + ikr}}{4\pi r}(\boldsymbol{n} \times -\hat{\boldsymbol{x}})(-ikR)(I_0/c) \tag{9}$$

$$= \frac{e^{-i\omega t + ikr}}{4\pi r} \cos\theta(-\hat{\boldsymbol{y}})(-ik\pi R)(I_0/c) \tag{10}$$

c) The radiated power is

$$\frac{dP}{d\Omega} = \frac{c}{2}\mathrm{Re}(r^2 \boldsymbol{n} \cdot (\boldsymbol{E} \times \boldsymbol{B}^*))\,. \tag{11}$$

With $\boldsymbol{E} = -\boldsymbol{n} \times \boldsymbol{B}$, we have

$$\boldsymbol{n} \cdot (-\boldsymbol{n} \times \boldsymbol{B}) \times \boldsymbol{B}^* = |\boldsymbol{B}|^2\,, \tag{12}$$

and

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{c}{2} r^2 |\boldsymbol{B}|^2 && (13) \\ &= \frac{c}{32\pi^2} \cos^2\theta\, (\pi k R I_0 / c)^2 && (14) \end{aligned}$$

It is perhaps useful to convert to MKS units:

$$\begin{aligned} \frac{I_0}{c} &\to \sqrt{\mu_o} I && (15) \\ c &\to \frac{1}{\sqrt{\mu_o \epsilon_o}} && (16) \end{aligned}$$

and using $\sqrt{\mu_o/\epsilon_o} = 376\,\mathrm{Ohm}$ we find

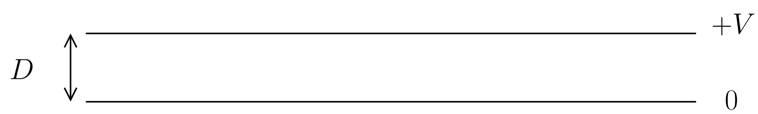
$$\frac{dP}{d\Omega} = 376\,\mathrm{Watts} \left(\frac{I_0}{\mathrm{amps}}\right)^2 \frac{(kR)^2}{32} \cos^2\theta \tag{17}$$

c) Since the magnetic field is in the $-\hat{\boldsymbol{y}}$ direction, for light propagating along the z axis the electric field is in the $-\hat{\boldsymbol{x}}$ direction, *i.e.* along the direction of the dipole moment.

Electromagnetism 3

Parameters of an electron tube

Consider an idealized electron tube (diode) consisting of infinite planar cathode and anode separated by a distance D in the z direction (see below). The cathode (at $z = 0$) may be regarded as an infinite supply of free electrons at rest. The anode (at $z = D$) is at potential $+V$ relative to the cathode. (V is sufficiently small that Newtonian physics applies.) The device is evacuated, so that only electrons are between the two electrodes. The current through such a device is determined by the flow of the charge of these electrons from the cathode ($z = 0$) to the anode ($z = D$).



a) (10 points) Use Poisson's equation, the equation of continuity, and the conservation of energy to derive a differential equation for the electric potential $\Phi(z)$ in steady state. Make sure you have the sign correct, and state the boundary conditions explicitly.

b) (6 points) Find $\Phi(z)$ and use it to determine the current density J as a function of the parameters of the problem and physical constants. *Hint:* Try a scaling solution of the form $\Phi(z) \propto z^\beta$.

c) (4 points) Put in numbers for a centimeter-sized device and an anode potential of 300 volts to *estimate* the impedance typical of electron tube circuits.

Solution

a) Let $v(z)$ = speed of electrons at distance c from the cathode.

Total energy of electron $= mv(z)^2/2 - e\Phi(z) = 0$, so $v(z) = (2e\Phi(z)/m)^{1/2}$.

Continuity: Current density $J = v(z)\rho(z)$ is constant, independent of z.

$$\text{Poisson: } \nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{1}$$

$$\frac{d^2\Phi}{dz^2} = +\frac{|J|}{\epsilon_0 v(z)} = \left[\frac{|J|}{\epsilon_0}\sqrt{\frac{m}{2e}}\right]\Phi(z)^{-1/2}. \tag{2}$$

Boundary conditions are $\Phi(0) = 0, \Phi(D) = V$. Note that there is no boundary condition on $\frac{d\Phi}{dz}$ at $z = 0$.

b) Hypothesize a solution of the form $\Phi(z) = Az^\beta$.

$$\frac{d^2\Phi}{dz^2} = A\beta(\beta-1)z^{\beta-2} = KA^{-1/2}\beta^{-1/2}. \tag{3}$$

(Here K is the factor in square brackets in equation 2.)

This works if $\beta - 2 = -1/2$, *i.e.*, $\beta = 4/3$ and $\frac{4}{9}A^{3/2} = K$. The solution is

$$\Phi(z) = \left(\frac{9J}{4\epsilon_0}\right)^{2/3}\left(\frac{m}{2e}\right)^{1/3} z^{4/3}. \tag{4}$$

The boundary condition that $\Phi(D) = V$ leads to

$$|J| = \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m}}\,\frac{V^{3/2}}{D^2}. \tag{5}$$

c) To substitute units, insert a factor of $c = 1/\sqrt{\mu_o\epsilon_o}$, note that

$$m_e c^2 = 0.5\,\text{MeV} = 0.5 \times 10^6\,\text{eV},$$

and recall that $\sqrt{\mu_o/\epsilon_o} = 376\,\Omega$. We find (using ($e \cdot V = 300\,\text{eV}$)

$$|J| = \frac{4}{9}\left(\frac{V}{376\,\Omega}\right)\frac{1}{D^2}\sqrt{\frac{2e \cdot V}{mc^2}} \tag{6}$$

$$= 121\,\frac{\text{Amps}}{\text{meter}^2}\left(\frac{V}{300\,\text{Volts}}\right)^{3/2}\left(\frac{\text{cm}}{D}\right)^2 \tag{7}$$

So taking the plate area to be $1\,\text{cm}^2$

$$\frac{V}{I} = 1.59 \times 376\,\Omega\sqrt{\frac{mc^2}{e \cdot V}}\,\frac{D^2}{\text{Area}} \tag{8}$$

$$= 25000\,\Omega\sqrt{\frac{300\,\text{Volts}}{V}}\left(\frac{D^2/\text{Area}}{1}\right) \tag{9}$$

Quantum Mechanics 1

Spin-$\frac{1}{2}$ resonance and neutron interferometry

I. An electron of charge e and mass m_e is subject to a uniform magnetic field $B_0\hat{z}$ and has its spin along the positive z-axis. At $t = 0$ an additional time-dependent magnetic field is switched on in the transverse plane with

$$B_\perp(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y} \tag{1}$$

a) (8 points) Write down the Schrödinger equation for this time-dependent problem and solve it.

b) (3 points) What is the probability in time to find the electron with its spin along the negative z-axis, and for what frequency is the spin flip maximum?

c) (3 points) Neutron spin flippers are based on this magnetic set-up. Denoting by t_n the time that a neutron is in the field, find the minimum value of t_n for a maximum spin flip to occur. Explicitly write down the neutron state at this time. The neutron magnetic moment is μ_n.

II. Now in a neutron interferometer, a neutron beam is split into two beams labeled (1) and (2). In beam (1) the neutrons are subject to the spin flipper with the time of flight adjusted for maximum spin flip. In beam (2) the neutrons acquire a spin-independent phase δ by a mechanism which we do not specify. There is no magnetic field acting on beam (2). Both beams are recombined and an interference pattern is observed. Let the initial neutron state at $t_i = 0$ be

$$\langle \vec{x}|\Psi_i\rangle = \frac{1}{\sqrt{2}}\left(\langle \vec{x}|\varphi_1\rangle|\uparrow\rangle + \langle \vec{x}|\varphi_2\rangle|\downarrow\rangle\right) \tag{2}$$

Here $\langle \vec{x}|\varphi_{1,2}\rangle$ are fixed and normalized spatial states along route (1) and (2) respectively.

d) (6 points) Determine the final neutron state $\langle \vec{x}|\Psi_f\rangle$ as the neutron beam recombines in the interferometer and calculate the expectation value of the neutron spin $\vec{\mathbf{S}}_n$ in this final state.

Solution

a. The Schrödinger equation in the spin-$\frac{1}{2}$ basis is

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = -\frac{e\hbar}{2m_e}\begin{pmatrix} B_0 & B_\perp e^{-i\omega t} \\ B_\perp e^{i\omega t} & -B_0 \end{pmatrix}|\Psi(t)\rangle \tag{3}$$

with the initial condition $|\Psi(0)\rangle = |\uparrow\rangle$. The solution is

$$|\Psi(t)\rangle = \begin{pmatrix} e^{-i\omega t/2}(\cos(\gamma t/2) + i((\omega - 2\omega_0)/\gamma)\sin(\gamma t/2) \\ -2ie^{i\omega t/2}(\omega_\perp/\gamma)\sin(\gamma t/2) \end{pmatrix} \tag{4}$$

with $\omega_0 = |e|B_0/2m_e$ and $\omega_\perp = |e|B_\perp/2m_e$ and

$$\gamma = \left((\omega - 2\omega_0)^2 + 4\omega_\perp^2\right)^{\frac{1}{2}} \tag{5}$$

b. The spin flip probability in time is

$$P_\downarrow(t) = \frac{4\omega_\perp^2}{(\omega - 2\omega_0)^2 + 4\omega_\perp^2}\sin^2\left(\frac{\gamma t}{2}\right) \tag{6}$$

The spin flip is resonant or maximum for $\omega = 2\omega_0$.

c. For the neutron $\omega_0 = \mu_n B_0/2$ and $\omega_\perp = \mu_n B_\perp/2$. The time of flight t_n for maximum spin flip is set by the resonance frequency at

$$P_\downarrow(t_n) = \sin^2(\omega_\perp t_n) = 1 \tag{7}$$

with a minimum value $t_n = \pi/2\omega_\perp$. The neutron spin state is

$$|\Psi(t_n)\rangle = -ie^{i\theta}|\downarrow\rangle \tag{8}$$

with $\theta = \omega_0 t_n = \pi B_0/2B_\perp$.

d. The final neutron state at the interference point is

$$|\Psi_f\rangle = \frac{e^{i\theta}}{\sqrt{2}} \left(-i|\varphi_1\rangle|\downarrow\rangle + e^{i(\delta-\theta)}|\varphi_2\rangle|\uparrow\rangle \right) \tag{9}$$

The averaged neutron spin $\vec{\mathbf{S}}_n = \frac{\hbar}{2}\vec{\mathbf{œ}}$ at the interference point is

$$\langle\Psi_f|\vec{\mathbf{S}}_n|\Psi_f\rangle = \frac{\hbar}{2}\langle\varphi_2|\varphi_1\rangle \left(\sin(\delta-\theta)\hat{\mathbf{x}} + \cos(\delta-\theta)\,\hat{\mathbf{y}}\right) \tag{10}$$

Quantum Mechanics 2

Pairs of Hamiltonians in one dimension

Consider a particle on the x-axis with potential $U(x)$ such that $U(x)$ vanishes as $x \to \pm\infty$, and $U(x)$ is everywhere negative and nonsingular. Recall that the ground state for such a system is always a nondegenerate bound state.

a) (4 points) Define $V(x) = U(x) - E_0$ where E_0 is the ground state energy. Write the Hamiltonian in factorized form as $H = A^\dagger A + E_0$ where $A = c\frac{d}{dx} + W(x)$ and c is a constant. Determine c and $W(x)$. (Hint: Express $V(x)$ in terms of the ground state wave function $\phi_0(x)$ and try the logarithmic derivative of $\phi_0(x)$ for W.)

b) (4 points) Show that A annihilates $\phi_0(x)$. Show that $H_1 = A^\dagger A + E_0$ and $H_2 = AA^\dagger + E_0$ have the same non-vanishing eigenvalues $E_n > E_0$. Draw a picture of the eigenvalues of H_1 and H_2, both the discrete and the continuous ones. (Hint: Act with A on H_1.)

c) (6 points) Consider now two systems, one with Hamiltonian $H_1 = A^\dagger A + E_0$ and another with Hamiltonian $H_2 = AA^\dagger + E_0$. Let $A^\dagger A = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1$ and $AA^\dagger = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2$. Construct the $W(x)$ which gives $V_1 = a^2$, where a is a constant, and construct the corresponding $V_2(x)$. Plot $V_2(x)$ and $V_1(x)$ as functions of x. (Hint: The solution of the Riccati equation $\frac{d}{dx}y + y^2 = 1$ is given by $y(x) = \tanh x$.)

d) (6 points) If $A^\dagger A$ has a constant potential $V_1 = a^2$, the solutions for H_1 are plane waves. Prove that then the potential $V_2(x)$ of H_2 is also reflectionless. (A potential is called reflectionless if every incoming plane wave of the continuous spectrum is transmitted without reflection. In other words, there is total transmission.)

Solution

a) Assuming $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U$ can be written as $H = A^\dagger A + E_0$ we get

$$
\begin{aligned}
H = A^\dagger A + E_0 &= \left(-c\tfrac{d}{dx} + W\right)\left(c\tfrac{d}{dx} + W\right) + E_0 \\
&= -c^2\tfrac{d^2}{dx^2} - c\left(\tfrac{dW}{dx}\right) + W^2 + E_0 \\
&= -\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2} + U\,.
\end{aligned}
$$

Clearly $c^2 = \frac{\hbar^2}{2m}$ and $-c\left(\frac{dW}{dx}\right) + W^2 = U - E_0 \equiv V$. Since $H\phi_0 = E_0\phi_0$ implies $-\frac{\hbar^2}{2m}\phi_0'' = (E_0 - U)\phi_0 = -V\phi_0$, we have $V = \frac{\hbar^2}{2m}\left(\frac{\phi_0''}{\phi_0}\right)$. We must now find the function W that yields V. As suggested, we try $W = \alpha\frac{\phi_0'}{\phi_0}$ where α is a constant to be determined. Then $-c\frac{dW}{dx} + W^2 = -\alpha c\left(\frac{\phi_0''}{\phi_0} - \left(\frac{\phi_0'}{\phi_0}\right)^2\right) + \alpha^2\left(\frac{\phi_0'}{\phi_0}\right)^2$. For $\alpha = -c$ we indeed get $c^2\frac{\phi_0''}{\phi_0} = V$. So

$$
c^2 = \frac{\hbar^2}{2m}; \quad W = -c\frac{\phi_0'}{\phi_0},
$$

and $H = -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + U$ can indeed be written as $H = A^\dagger A + E_0$ where

$$
A = \sqrt{\frac{\hbar^2}{2m}}\left(\frac{d}{dx} - \frac{\phi_0'}{\phi_0}\right).
$$

b) The operator A annihilates the ground state as is clear from

$$
A\phi_0 = \left(c\frac{d}{dx} + W\right)\phi_0 = \left[c\frac{d}{dx} - c\left(\frac{\phi_0'}{\phi_0}\right)\right]\phi_0 = 0\,.
$$

So $H_1\phi_0 = \left(A^\dagger A + E_0\right)\phi_0 = E_0\phi_0$. If $H_1\phi_n = E_n\phi_n$ for $n > 0$, then acting with A on this equation, we get $AA^\dagger(A\phi_n) = (E_n - E_0)(A\phi_n)$ and $A\phi_n$ is nonzero because $A^\dagger A\phi_n = (E_n - E_0)\phi_n \neq 0$. ($\phi_n$ and $E_n - E_0$ are both nonvanishing.) So then $H_2\tilde{\phi}_n = E_n\tilde{\phi}_n$ where $\tilde{\phi}_n = A\phi_n$. Conversely, if $H_2\tilde{\phi}_n = E_n\tilde{\phi}_n$ for $n > 0$, then

$$
A^\dagger H_2\tilde{\phi}_n = A^\dagger A(A^\dagger\tilde{\phi}_n) + E_0A^\dagger\tilde{\phi}_n = E_nA^\dagger\tilde{\phi}_n\,,
$$

so then $H_1\phi_n = E_n\phi_n$ with $\phi_n = A^\dagger\tilde{\phi}_n$. In a picture

$$
\begin{array}{cccccl}
 & H_1 & & H_2 & & \\
 & \equiv & \leftrightarrow & \equiv & \} & \text{continuous spectrum} \\
 & \vdots & & \vdots & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} & \\
E_2 : & \phi_2 = A^\dagger \tilde{\phi}_2 & \leftrightarrow & A\phi_2 = \tilde{\phi}_2 & & \\
E_1 : & \phi_1 = A^\dagger \tilde{\phi}_1 & \leftrightarrow & A\phi_1 = \tilde{\phi}_1 & & \text{bound states} \\
E_0 : & \phi_0 & & - & &
\end{array}
$$

So H_1 and H_2 form conjugate pairs, and they have the same spectrum except that one of them has a normalizable ground state with energy E_0, while the other has no eigenvalue E_0.

c) We consider now a special case: $V_1 = a^2$. Recall

$$
\begin{aligned}
H_1 &= \left(-c\tfrac{d}{dx} + W\right)\left(c\tfrac{d}{dx} + W\right) + E_0 = -\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2} + U_1 \\
H_2 &= \left(c\tfrac{d}{dx} + W\right)\left(-c\tfrac{d}{dx} + W\right) + E_0 = -\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2} + U_2
\end{aligned}
$$

$$
\left.\begin{aligned}
V_1 &= -c\tfrac{dW}{dx} + W^2 \\
V_2 &= c\tfrac{dW}{dx} + W^2
\end{aligned}\right\}
\qquad
\begin{aligned}
V_2 + V_1 &= 2W^2 \\
V_2 - V_1 &= 2c\tfrac{dW}{dx}\,.
\end{aligned}
$$

If $V_1 = a^2$ then $-c\frac{dW}{dx} + W^2 = a^2$, so

$$
-\tfrac{c}{a^2}\tfrac{dW}{dx} + \tfrac{W^2}{a^2} = 1 \Rightarrow \tfrac{W}{a} = -y\,, \quad \frac{ax}{c} = z\,.
$$

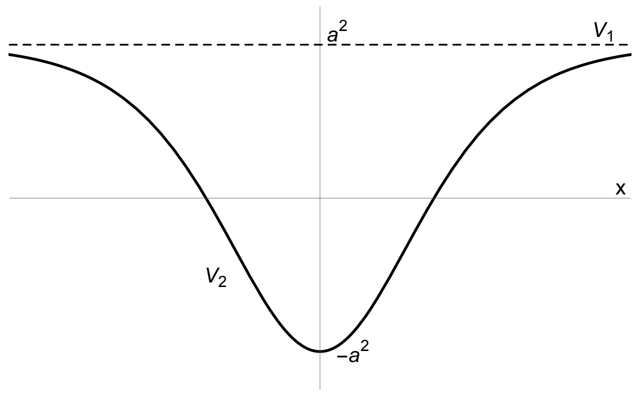
Then $\frac{d}{dz}y + y^2 = 1 \Rightarrow y(z) = \tanh z$. So $W(x) = -a\tanh\left(\frac{ax}{c}\right)$. Then

$$
V_2 = 2W^2 - V_1 = 2a^2\tanh^2\left(\frac{ax}{c}\right) - a^2 = \left[1 - \frac{2}{\cosh^2\left(\frac{ax}{c}\right)}\right]a^2\,.
$$

Clearly, H_1 has only a continuous spectrum, but H_2 has both a continuous spectrum and at least one (in fact, precisely one) bound state. So, $H_1(H_2)$ here corresponds to $H_2(H_1)$ in part a).

d) The eigenfunctions of H_1 with $V_1 = a^2$ are plane waves $\phi_1(k) = e^{i(kx-\omega t)}$, and the eigenfunctions of H_2 are obtained as $\phi_2(k) = A\phi_1(k)$. Since $A = \left(c\frac{d}{dx} + W\right)$ we get, using $W = -a\tanh\frac{ax}{c}$,

$$
\begin{aligned}
\phi_2(k) &= \left(i\,c\,k - a\tanh\frac{ax}{c}\right)e^{i(kx-\omega t)} \\
&\xrightarrow{x\to\pm\infty} (i\,c\,k \mp a)e^{i(kx-\omega t)} = \sqrt{(ck)^2 + a^2}\,e^{i(kx-\omega t\pm\delta/2)}
\end{aligned}
$$



where δ is the phase shift.

The crucial point is that there are no terms with $e^{-ikx-i\omega t}$ for $x \to -\infty$. This means there is no reflection.

Comment: There is an infinite set of pairs with potentials $V_l = \frac{l(l+1)}{\cosh^2(\frac{ax}{c})} a^2$. We studied the case $l = 1$, which corresponds to the sine-Gordon soliton. The case $l = 2$ corresponds to the kink solution. They are all reflectionless. If one adds a two-component fermion with a Yukawa term such that the action becomes supersymmetric, then after iteration H_1 is the Hamiltonian for one component and H_2 is the Hamiltonian for the other component. Then the zero-point energies of the boson cancel the zero-point energies of the fermion (the "cosmogical constant" vanishes). Because H_1 and H_2 are "isospectral" one can calculate various quantities in closed form. The theory of such systems with conjugate Hamiltonians is called supersymmetric quantum mechanics.

Quantum Mechanics 3

Shell model for atomic nuclei

Atomic nuclei can be described by the shell-model which consists of spin-1/2 protons and neutrons filling the states of a spherically symmetric potential $V(r)$. A simple approximation is $V(r) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$, with energy levels of the form

$$E_N = \left(N + \frac{3}{2}\right)\hbar\omega \tag{1}$$

a) (2 points) What is the degeneracy of E_N as a function of N (including spin)? As a check, list the single-particle states within the first 3 shells $N = 0, 1, 2$.

b) (6 points) We introduce a radial quantum number n and orbital quantum number l through $N = 2(n-1) + l$. Write the explicit expressions for the energies, parities and degeneracies (including spin) for the first 5 shells. Identify the magic numbers corresponding to closed shells by analogy with the noble gases.

c) (6 points) The spin-orbit interaction between the nucleons

$$V_{so} = a\,\vec{s}\cdot\vec{l} \quad , \quad a < 0 \tag{2}$$

splits the states within a given shell. Calculate the first-order energy shifts caused by V_{so} and the corresponding degeneracies for the first 5 shells.

d) (3 points) O^{17} consists of 8 protons and 9 neutrons. Identify its spectroscopic nl_j assignment (where $\vec{j} = \vec{l} + \vec{s}$), and find its energy shift and parity.

e) (3 points) Nuclear data indicate that the fourth magic number is 50, in disagreement with what you found in **b**. An intruder state from the 5th shell got into the 4th shell. What are the quantum numbers of the intruder state?

Solution

a. $N = n_x + n_y + n_z$ with each $n_i = 0, 1, ...$ For fixed N and n_x the degeneracy is $N - n_x + 1$. Thus, including a factor 2 for spin, the degeneracy of the Nth level is

$$d(N) = 2 \times \sum_{n_x=0}^{N} (N - n_x + 1) = (N+1)(N+2) \tag{3}$$

For the $N = 0, 1, 2$ wave functions we define $a^2 = \hbar/(m\omega)$ with H_n the nth Hermite polynomial

$$\begin{aligned}
&N = 0 \quad n_x = 0, n_y = 0, n_z = 0 \quad \Psi_{0,000}(\vec{x}) \approx e^{-\frac{1}{2a^2}(x^2+y^2+z^2)} \chi_s \\
&N = 1 \quad n_x = 1, n_y = 0, n_z = 0 \quad \Psi_{0,100}(\vec{x}) \approx H_1(x) \Psi_{0,000}(\vec{x}) \text{ etc.} \\
&N = 2 \quad n_x = 2, n_y = 0, n_z = 0 \quad \Psi_{0,200}(\vec{x}) \approx H_2(x) \Psi_{0,000}(\vec{x}) \text{ etc.}
\end{aligned} \tag{4}$$

The degeneracies are $d(0) = 2, d(1) = 6$ and $d(2) = 12$.

b. Each shell follows by summing over the available $2(2l + 1)$

$$\begin{aligned}
&n = 1, l = 0 \;\; 1s \;\; E_0 = \frac{3}{2}\hbar\omega \;\; P = + \;\; d_0 = 2d(0) = 2 \;\; N_{\text{tot}} = 2 \\
&n = 1, l = 1 \;\; 1p \;\; E_1 = \frac{5}{2}\hbar\omega \;\; P = - \;\; d_1 = 2d(1) = 6 \;\; N_{\text{tot}} = 2 + 6 = 8 \\
&n = 2, l = 0 \; n = 1, l = 2, \;\; 2s, 1d \;\; E_2 = \frac{7}{2}\hbar\omega \;\; P = + \;\; d_2 = 2d(2) = 12 \;\; N_{\text{tot}} = 12 + 8 = 20 \\
&n = 2, l = 1, n = 1, l = 3 \;\; 2p, 1f \;\; E_3 = \frac{9}{2}\hbar\omega \;\; P = - \;\; d_3 = 2d(3) = 20 \;\; N_{\text{tot}} = 20 + 20 = 40 \\
&n = 1, l = 4, n = 2, l = 2, n = 3, l = 0 \;\; 1g, 2d, 3s \;\; E_4 = \frac{11}{2}\hbar\omega \;\; P = + \;\; d_4 = 30 \;\; N_{\text{tot}} = 70
\end{aligned} \tag{5}$$

c. The total angular momentum $\vec{j} = \vec{l} + \vec{s}$. Thus

$$V_{so} = \frac{a\hbar^2}{2} \left(j(j+1) - l(l+1) - \frac{3}{4} \right) \tag{6}$$

with $j = l \pm \frac{1}{2}$ accessible. Since $a < 0$ the larger j states are pushed down. The energy shifts follow through the nl_j assignments of each state.

d. O^{17} is doubly magic O^{16} plus a neutron. It should be the lowest state in the third shell which is $1d_{5/2}$ with $P = +$ and $j = (2 + \frac{1}{2})\hbar$ and a spin-orbit splitting

$$\Delta E_{so} = \frac{a\hbar^2}{2}\left((2+\frac{1}{2})(2+\frac{3}{2}) - 2(2+1) - \frac{3}{4}\right) = a\hbar^2 \tag{7}$$

e. The intruder state has the largest j and is thus $1g_{9/2}$ with $P = +$ and 10 states. Thus the observed magic sequence is $2, 8, 20, (40 + 10 = 50)$ instead of $2, 8, 20, 40$ without spin-orbit.

Statistical Mechanics 1

Thermodynamic properties of a ferroelectric crystal

Consider a ferroelectric system of N molecules in zero electric field $\mathcal{E} = 0$. Each molecule has two energy states available, of energy $-\frac{1}{2}\kappa$ and $+\frac{1}{2}\kappa$ respectively, where κ is a constant. If l molecules are in the excited state, then the energy of the crystal is given by

$$E^0_{N,l} = -\frac{1}{2}(N-l)\,\kappa + \frac{1}{2}\,l\,\kappa\,, \quad l = 0, 1, 2, ...N, \tag{1}$$

with degeneracy

$$g_{N,l} = 2\frac{N!}{(N-l)!\,l!}\,. \tag{2}$$

In the presence of a non-zero electric field $\mathcal{E}$ the degeneracies are partially split by the energy-level changes:

$$\Delta E_{N,l} = \pm\,\nu\,l\mathcal{E}, \tag{3}$$

where ν is the electric moment of the molecules that couples to the external electric field.

a) (3 points) What is the ground state energy per molecule $u_0(\mathcal{E})$ at $T = 0$, as a function of the electric field $\mathcal{E}$?

b) (2 points) Calculate the partition function $Q_N(T, \mathcal{E})$ and the free energy per molecule of this system *.

c) (7 points) Find the internal energy per molecule $u(T, \mathcal{E})$ at finite temperature T and electric field $\mathcal{E}$, and show that, in the limit $T \to 0$, $u(T, \mathcal{E})$ approaches $u_0(\mathcal{E})$. Find $u(T, \mathcal{E})$ as a function of $\mathcal{E}$ in the limit $T \to \infty$.

d) (8 points) Find the entropy per molecule, $s(T, \mathcal{E})$, for the two limits of the temperature ($T \to \infty$ and $T = 0$), paying special attention to the field values $\mathcal{E} = \pm\frac{\kappa}{\nu}$. Relate your results to the Third Law of Thermodynamics.

---

*Hint: $(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k$

Solution

a) For $\mathcal{E} = 0$, the ground state is $l = 0$ such that $E_{N,0} = -\frac{1}{2}N\kappa$, and therefore

$$u_0(\mathcal{E}) \equiv E/N = -\frac{1}{2}\kappa. \tag{4}$$

For small $\mathcal{E}$, the ground state remains unchanged, but for $|\mathcal{E}| > \kappa/\nu$, $l = N$ becomes the ground state, and

$$u_0(\mathcal{E}) = E/N = \left(-\frac{1}{2}N\kappa + N(\kappa - \nu|\mathcal{E}|)\right)/N = \left(\frac{1}{2}\kappa - \nu|\mathcal{E}|\right) \tag{5}$$

b) The partition function is given by

$$\begin{aligned} Q &= \sum_{\text{states}} e^{-\beta E_{\text{state}}} && (6) \\ &= \sum_l \frac{g_{N,l}}{2} e^{-\beta(E^0_{N,l} + \nu l \mathcal{E})} + \sum_l \frac{g_{N,l}}{2} e^{-\beta(E^0_{N,l} - \nu l \mathcal{E})} \\ &= e^{\beta \frac{N}{2}\kappa} \sum_l \frac{N!}{(N-l)!\,l!} \left[e^{-\beta(\kappa+\nu\mathcal{E})l} + e^{-\beta(\kappa-\nu\mathcal{E})l}\right] \\ &= e^{\beta \frac{N}{2}\kappa} \left[(1 + e^{-\beta(\kappa+\nu\mathcal{E})})^N + (1 + e^{-\beta(\kappa-\nu\mathcal{E})})^N\right] && (7) \end{aligned}$$

and therefore the free energy

$$\begin{aligned} A &= -kT \ln Q && (8) \\ &= -\frac{N}{2}\kappa - kT \ln\left[(1 + e^{-\beta(\kappa+\nu\mathcal{E})})^N + (1 + e^{-\beta(\kappa-\nu\mathcal{E})})^N\right] && (9) \end{aligned}$$

c) The internal energy per molecule is

$$\begin{aligned} u(T, \mathcal{E}) &= -\frac{\partial \ln Q}{\partial \beta} \times \frac{1}{N} && (10) \\ &= -\frac{\kappa}{2} + \frac{1}{[1 + e^{-\beta(\kappa+\nu\mathcal{E})}]^N + [1 + e^{-\beta(\kappa-\nu\mathcal{E})}]^N} \times \\ &\quad (e^{-(\kappa+\nu\mathcal{E})}(\kappa + \nu\mathcal{E})[1 + e^{-\beta(\kappa+\nu\mathcal{E})}]^{N-1} \\ &\quad + e^{-(\kappa-\nu\mathcal{E})}(\kappa - \nu\mathcal{E})[1 + e^{-\beta(\kappa-\nu\mathcal{E})}]^{N-1}) && (11) \end{aligned}$$

Now as $T \to 0$ we have $\beta \to \infty$. For small $\mathcal{E}$, this means

$$u(T, |\mathcal{E}| < \frac{\kappa}{\nu}) = -\kappa/2, \tag{12}$$

whereas for $\mathcal{E} > \kappa/\nu$ we have $e^{-\beta(\kappa-\nu\mathcal{E})} \to \infty$ but $e^{-\beta(\kappa+\nu\mathcal{E})} \to 0$, so

$$u(T, \mathcal{E} > \frac{\kappa}{\nu}) = -\frac{\kappa}{2} + (\kappa - \nu\mathcal{E}), \tag{13}$$

and for $\mathcal{E} < -\kappa/\nu$,

$$u(T, \mathcal{E} < -\frac{\kappa}{\nu}) = -\frac{\kappa}{2} + (\kappa + \nu\mathcal{E}) \tag{14}$$

which proves that indeed $u(T, \mathcal{E}) \to u_0(\mathcal{E})$ for $T \to 0$.

On the other hand as $T \to \infty$, we have $\beta \to 0$ such that $e^{-\beta\alpha} \to 1 + \mathcal{O}(\beta\alpha)$. This means that

$$\begin{aligned} u(T, \mathcal{E}) &= -\frac{\kappa}{2} + \left[\frac{1 \cdot (\kappa + \nu\mathcal{E}) \cdot 2^{N-1} + 1 \cdot (\kappa - \nu\mathcal{E}) \cdot 2^{N-1}}{2^N + 2^N}\right] &(15)\\ &= -\frac{\kappa}{2} + \frac{\kappa}{2} = 0 + \mathcal{O}(\beta) &(16)\end{aligned}$$

d) We have

$$\begin{aligned} S &= -\frac{\partial A}{\partial T} = \frac{\partial}{\partial T}(kT \ln Q) \\ &= k \ln Q - k\beta \frac{\partial}{\partial \beta} \ln Q \\ &= k \ln Q + k\beta U &(17)\end{aligned}$$

As $T \to \infty$ we we have $\beta \to 0$ and the second term in S is negligible. Hence

$$\begin{aligned} S &= k\left[\beta \frac{N}{2}\kappa + \ln\left[(1 + e^{-\beta(\kappa+\nu\mathcal{E})})^N + (1 + e^{-\beta(\kappa-\nu\mathcal{E})})^N\right]\right] \\ &= k\left[\beta \frac{N}{2}\kappa + \ln\left[2^N(1 - N\frac{\beta}{2}(\kappa + \nu\mathcal{E}) + \dots) + 2^N(1 - N\frac{\beta}{2}(\kappa - \nu\mathcal{E}) + \dots)\right]\right] \\ &= (N+1)k \ln 2 + \mathcal{O}(\beta, \mathcal{E}), &(18)\end{aligned}$$

and therefore the entropy per molecule is

$$s \equiv \frac{S}{N} \approx k \ln 2 \tag{19}$$

As $T \to 0$, for small $\mathcal{E}$ we have $e^{\beta(\kappa \pm \nu\mathcal{E})} \to 0$. So with $U = -N\frac{\kappa}{2}$ and $\ln Q = \beta\frac{N\kappa}{2} + \ln 2$, we get

$$\begin{aligned} S &= k \ln Q + k\beta U \\ &= k \ln 2 + k\left(\beta\frac{N\kappa}{2} - \beta\frac{N\kappa}{2}\right) \\ &= k \ln 2, &(20)\end{aligned}$$

which means that

$$s = \frac{S}{N} \to 0, \tag{21}$$

while for $\mathcal{E} > \kappa/\nu$ we have $e^{-\beta(\kappa-\nu\mathcal{E})} \to \infty$, but $e^{-\beta(\kappa+\nu\mathcal{E})} \to \infty$, and $U = N\left[-\frac{\kappa}{2} + (\kappa - \nu\mathcal{E})\right]$. So

$$\begin{aligned} \ln Q &= \beta\frac{N\kappa}{2} - N\beta(\kappa - \nu\mathcal{E}) && (22) \\ S &= k\ln Q + k\beta U = 0 && (23) \end{aligned}$$

and hence

$$s = \frac{S}{N} = 0 \tag{24}$$

Similarly for $\mathcal{E} < -\kappa/\nu$, where also $s = 0$.

Exactly at $\mathcal{E} = \kappa/\nu$ (or $\mathcal{E} = -\kappa/\nu$), we have $U = -N\kappa/2$ and $e^{-\beta(\kappa-\nu\mathcal{E})} = 1$, so

$$\ln Q = \beta\frac{N}{2}\kappa + \ln(2^N) = \beta\frac{N}{2}\kappa + N\ln 2 \tag{25}$$

and

$$S = k\ln Q + k\beta U = kN\ln 2; \qquad s = \frac{S}{N} = k\ln 2 \tag{26}$$

Note that these results are really just saying that

$$S_{T=0} = k\ln\left(\text{degeneracy of ground state}\right) \tag{27}$$

For small $\mathcal{E}$ with $|\mathcal{E}| < \kappa/\nu$, the degeneracy of the ground state $= 2$, whereas for $|\mathcal{E}| > \kappa/\nu$, the degeneracy $= 1$. But for $\mathcal{E} = \kappa/nu$, the degeneracy $\approx 2^N$.

The Third Law tells us that $S_{T\to 0} = 0$, which we do not find here. However, in any real physical system this degeneracy will be broken, so that S goes to 0.

Statistical Mechanics 2

Two-dimensional, nonrelativistic Bose gas

A two-dimensional ideal, spinless and nonrelativistic Bose gas is maintained in an area A with finite temperature T and chemical potential μ, with $z = e^{\mu/k_B T}$.

a) (8 points) Calculate the grand partition function $\mathbf{Z}(z, A, T)$. Separate the zero momentum part.

b) (5 points) Calculate the average density of bosons $\mathbf{n}(z, A, T)$. Show that z must be less than 1 for any density.

c) (2 points) Can the two-dimensional idea gas Bose condense? Explain.

d) (5 points) What changes in these arguments in three dimensions, and why?†

Solution

a. For an ideal Bose gas

$$\mathbf{Z}(z, A, T) = \prod_p \left(1 - z e^{-p^2/(2mk_B T)}\right)^{-1} \tag{1}$$

Define the squared thermal wavelength $\lambda^2 = (2\pi\hbar^2)/(mk_B T)$ and use the given formula to perform the p-integration

$$\begin{aligned} \frac{\ln \mathbf{Z}}{A} &= -\frac{1}{A}\sum_p \ln\left(1 - z e^{-p^2/(2mk_B T)}\right) \\ &= -\frac{1}{A}\ln(1-z) + \frac{1}{\lambda^2}\sum_{k=1}^{\infty}\frac{z^k}{k^2} \end{aligned} \tag{2}$$

b. The number of bosons per unit area is

†Hint: for nonzero momenta you may use the following expansion to do the integrals: $\ln(1/(1-x)) = \sum_{k=1}^{\infty} \frac{x^k}{k}$

$$\mathbf{n}(z, A, T) = \frac{1}{A}\frac{\partial \ln \mathbf{Z}}{\partial \ln z} = \frac{1}{A}\frac{z}{1-z} + \frac{1}{\lambda^2}\sum_{k=1}^{\infty}\frac{z^k}{k} \tag{3}$$

with the first contribution from the zero-momentum state is separated.

c. For fixed density $\mathbf{n}$, Bose-condensation amounts to a finite density of bosons $\mathbf{n}$ in the ground state which is the first contribution in (2). This is possible only if $z \to 1 - \#/A < 1$ for any density. Thus

$$\frac{1}{A}\frac{z}{1-z} \approx \mathbf{n} - \frac{1}{\lambda^2}\sum_{k=1}^{\infty}\frac{1}{k} \tag{4}$$

The second contribution diverges implying no finite condensate in 2-dimension.

d. In 3-dimension the arguments are un-changed except for the substitution

$$\sum_{k=1}^{\infty}\frac{1}{k} \to \sum_{k=1}^{\infty}\frac{1}{k^2} \tag{5}$$

in (4). The RHS is now finite. Bose-Einstein condensation of an ideal Bose gas is allowed in 3-dimensions.

Statistical Mechanics 3

Mean-field theory of an Ising-type model in a transverse field

Consider the system of spins-1/2 on a lattice with q nearest-neighbors for each site, characterized by the Hamiltonian

$$H = -J\Big[\sum_{\langle i,j\rangle} \sigma_z^{(i)}\sigma_z^{(j)} + g\sum_i \sigma_x^{(i)}\Big]\,,$$

in equilibrium at temperature T. Here $\sigma_z^{(i)}$, $\sigma_x^{(i)}$ are the Pauli matrices of the ith spin, and the sum in the first term is taken over the pairs of the nearest-neighbor spins. As usual, the dimensionality of the lattice does not affect the results in the mean-field approximation.

a) (6 points) Adopting the standard mean-field approach, introduce the effective single-spin Hamiltonian H_0 and calculate the thermal averages

$$m_z = \langle \sigma_z^{(i)} \rangle\,, \quad m_x = \langle \sigma_x^{(i)} \rangle\,.$$

b) (6 points) Imposing the relevant self-consistency condition, find the transition temperature T_c of the temperature-driven phase transition from the paramagnetic (vanishing m_z) to the ferromagnetic (non-vanishing m_z) state, and determine the range of the parameter g, when such a phase transition exists. What is the expression for T_c in the limit $q \to 0$?

c) (8 points) In the situation with vanishing temperature, $T = 0$, analyze m_z as a function of g from the same self-consistency condition. Determine the point g_c of a "quantum phase transition" into a ferromagnetic state, and find m_z and m_x as functions of the small difference $\delta g = g_c - g$.

Solution

(a) As usual in the mean-field approximation, the effective single-spin Hamiltonian H_0 is obtained by replacing the spins interacting with it, by their average values. In this way, H_0 is obtained as:

$$H_0 = -J \begin{pmatrix} qm_z & g \\ g & -qm_z \end{pmatrix}.$$

Equilibrium state of this Hamiltonian gives for the spin averages:

$$m_z = \frac{1}{Z}\mathrm{Tr}\{\sigma_z e^{-\beta H_0}\}, \quad m_x = \frac{1}{Z}\mathrm{Tr}\{\sigma_x e^{-\beta H_0}\}, \quad \text{where } Z = \mathrm{Tr}\{e^{-\beta H_0}\}, \ \beta \equiv 1/k_B T.$$

One way of calculating the averages explicitly is to use the standard formula for the exponent of Pauli matrices:

$$e^{-\beta H_0} = e^{\beta J(qm_z\sigma_z + g\sigma_x)} = \cosh\beta J\Omega + \frac{\sinh\beta J\Omega}{\Omega}(qm_z\sigma_z + g\sigma_x), \quad \Omega \equiv \left[(qm_z)^2 + g^2\right]^{1/2}.$$

From this, we see that

$$Z = 2\cosh\beta J\Omega,$$

and finally,

$$m_z = \frac{qm_z}{\Omega}\tanh\beta J\Omega, \quad m_x = \frac{g}{\Omega}\tanh\beta J\Omega.$$

(b) Equation for m_z obtained in part (a) provides the mean-field self-consistency condition. At $T > T_c$, this condition is satisfied by vanishing z-component of magnetization, $m_z = 0$. It becomes non-vanishing if

$$1 = \frac{q}{\Omega}\tanh\beta J\Omega.$$

Solving this equation for β we find:

$$\frac{1}{k_B T} = \frac{1}{2J\Omega}\ln\left[\frac{q+\Omega}{q-\Omega}\right].$$

To find the critical temperature T_c, we take $m_z = 0$, i.e. $\Omega = g$, in this relation and get:

$$T_c = \frac{J}{k_B}\frac{2g}{\ln[(q+g)/(q-g)]}.$$

From this relation, we see that non-vanishing transition temperature T_c exists only if $g < q$. It also shows that in the limit $q \to 0$, T_c has the same mean-filed value as for the regular Ising model:

$$T_c = \frac{Jq}{k_B}.$$

(c) For vanishing temperature, $T = 0$, the self-consistency condition takes the form

$$m_z = \frac{q m_z}{[(q m_z)^2 + g^2]^{1/2}},$$

and can be solved for m_z explicitly:

$$m_z = [1 - g^2/q^2]^{1/2}.$$

As one can see from this equation, the non-zero solution exists only for $g < g_c = q$. For $g > g_c$, only the vanishing solution, $m_z = 0$, is possible. In the vicinity of this "quantum phase transition", when $\delta g = g_c - g \ll g_c$, m_z can be written as

$$m_z = \begin{cases} [2\delta g/g_c]^{1/2}, & g < g_c\,, \\ 0\,, & g > g_c\,. \end{cases}$$

This square-root dependence on δg coincides with the mean-field result for the order parameter in the temeprature-driven phase transition.

To find m_x, one can notice that at $T = 0$, components of the magnetization found in part (a) satisfy the natural condition

$$m_z^2 + m_x^2 = 1\,.$$

From this relation,

$$m_x = \begin{cases} [1 - 2\delta g/g_c]^{1/2}, & g < g_c\,, \\ 1\,, & g > g_c\,. \end{cases}$$