Qualifying Exam (May 2023): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10,11,12

All problems are weighted equally. On this cover page write which seven problems you want graded.

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

(1). Consider the following (knapsack) LP:

$$\begin{array}{rcl} \max\ z & = & 150x_1 + 100x_2 + 99x_3 \\ & & 51x_1 + 50x_2 + 50x_3 \quad \leq \quad 100 \\ & & x_1, x_2, x_3 \quad \geq \quad 0 \\ & & x_1, x_2, x_3 \quad \text{integer} \end{array}$$

Solve this problem by first finding the LP optimum and then finding the integer optimum by the cutting plane method.

(2). Consider the LP

$$\begin{array}{rcll} \max\ z & = & \sum_{j=1}^{n} p_j x_j & \\ \text{s.t.} & & \sum_{j=1}^{n} q_j x_j \leq \beta & \\ & & x_j \leq 1, & j = 1, 2, \ldots, n \\ & & x_j \geq 0, & j = 1, 2, \ldots, n \end{array}$$

Here, the numbers $p_j, j = 1, 2, \ldots, n$ are positive and sum to one, and the same is true for the q_j: $\sum_{j=1}^{n} q_j = 1, q_j > 0$. Furthermore, assume that

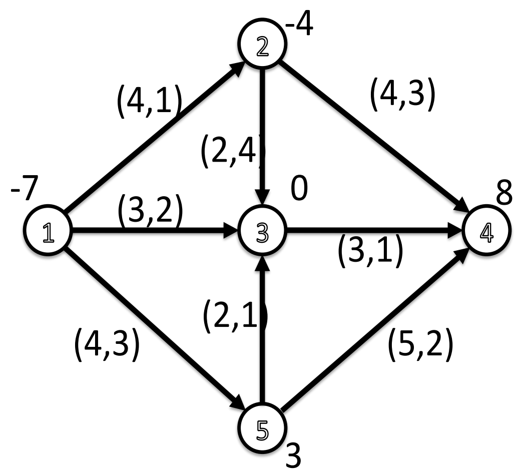
$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n}$$

and that the parameter β is a (small) positive number. Let $k = \min \{j : q_{j+1} + \cdots + q_n \leq \beta\}$. Let y_0 denote the dual variable associated with the constraint involving β, and let y_j denote the dual variable associated with the upper bound of 1 on variable x_j. Using duality theory, show that the optimal values of the primal and dual variables are given by

$$x_j = \begin{cases} 0, & j < k \\ \dfrac{\beta - q_{k+1} - \cdots - q_n}{q_k}, & j = k \\ 1, & j > k \end{cases}$$

$$y_j = \begin{cases} \dfrac{p_k}{q_k}, & j = 0 \\ 0, & 0 < j \leq k \\ q_j \left(\dfrac{p_j}{q_j} - \dfrac{p_k}{q_k} \right), & j > k \end{cases}$$

(3). We want to solve the following minimum cost network flow problem (here I am following the Chvatal notation from class, $b_i < 0$ is a supply etc. The pair of numbers next to each arc are (u_{ij}, c_{ij}), the upper bound and the cost for that arc. Numbers next to each node are supply/demand at that node.)



Using vertex 1 as the root and with artificial arc 14 still present, phase I has resulted in the BFS:

$$(x_{12}, x_{13}, x_{14}, x_{15}) = (0, 0, 4, 3)\ ,\ x_{24} = 4 = u_{24}\ .$$

1. Using upper bounded network simplex complete the phase I process to eliminate artificial arc 14 and arrive at a genuine BFS. Give flow values along edges and fair prices at vertices.
2. Is the BFS arrived at in part (a) the optimal solution? If yes justify by verifying that it satisfies the optimality conditions. If not, use the upper bounded network simplex method to find the optimal solution.

(4). Consider a two-server queueing system. The service times at the two servers are independent exponential random variables with parameters λ and μ, respectively. You arrive at a time when both servers are busy but there is no one else waiting in line. (1) Suppose you randomly pick a server, wait for that server to become free and then join the service. What is the probability p that you are the last of the three customers to leave the system. (2) If you pick the server that becomes free first, find the probability p.

(5). Let $\{Z_n\}$ be a DTMC with state space S and transition matrix $P = [p_{i,j}]$ satisfying $p_{i,i} = 0$ for all $i \in S$. Let $\{N(t)\}$ be a Poisson process with intensity $\lambda > 0$ that is independent of $\{Z_n\}$. Define a continuous time process $Z(t) = Z_{N(t)}$. Show that $\{Z(t)\}$ is a CTMC (write down its transition rate matrix Q) and compute its transition probabilities $P_{i,j}(t) = P(Z(t) = j|Z(0) = i)$ for $i, j \in S$.

(6). Consider a machine that processes jobs. Jobs arrive according to a Poisson process with rate λ, and service times of successive jobs are i.i.d. exponentially distributed with mean $1/\mu$. When there are no jobs, the machine is turned off. As soon as a new job arrives, the machine is turned on. The time to turn on the machine is an exponential random variable with mean $1/\theta$. Assuming $\lambda/\mu < 1$, compute the long run average number of jobs in the system and the mean waiting in system of a job.

(7). Let $Z \sim N(0,1)$ be a standard normal random variable. Using a majorizing function of the form $t(x) = ce^{-x}$ (c is a constant), give an acceptance-rejection algorithm for generating random variates from the distribution of $|Z|$. Describe how to obtain a random observation from Z based on a random realization of $|Z|$.

(8). Give an algorithm for generating random variates from the following cumulative distribution function

$$F(x) = \begin{cases} \frac{1-e^{-2x}+2x}{3}, & \text{if } 0 < x \leq 1 \\ \frac{3-e^{-2x}}{3}, & \text{if } 1 < x < \infty. \end{cases}$$

(9). Consider a simple n-gon P and a simple n-gon Q in the plane. As usual, we consider each to be a "solid", closed region in the plane, including the boundary and the interior.

(a). Assume that P and Q are disjoint. Our goal is to find a segment uv, if it exists, that joins a vertex u of P to a vertex v of Q such that the interior of the segment uv is disjoint from P, Q (the only places uv intersects the polygons is at the endpoints u, v). Is this always possible? Describe an efficient algorithm to determine such a segment uv (if one exists); what is the running time (in big-Oh)?

(b). Now suppose we want to determine if the polygons intersect of not (i.e., is $P \cap Q = \emptyset$?). Explain how this can be done efficiently using the best methods you know. Give the running time (in big-Oh) and justify your answer.

(c). Now suppose we want to compute $CH(P \cup Q)$. How efficiently can this be done? Justify.

(10). Let S be a set of n points in the plane. A "pinned empty disk" is a circular disk, D, whose interior contains no points of S and whose boundary contains at least 3 points of S.

(a). How many pinned empty disks can there be? Give the best upper bound you can, and justify it.

(b). How efficiently can you compute all pinned empty disks? Justify.

(c). How efficiently can you compute any *one* pinned empty disk? Explain.

(d). Suppose you want to find the convex hull of all pinned empty disks. (The input is just the n points S, in no particular order, and the output should be a boundary description of the convex hull of all pinned empty disks.) Explain how this convex hull can be computed efficiently.

(11). Let $X \in L^1(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$, $\mathcal{H}$ be sub σ-algebras of $\mathcal{F}$. Moreover let $\mathcal{H}$ be independent of $\sigma(\sigma(X), \mathcal{G})$. Show that $E\{X|\sigma(\mathcal{G}, \mathcal{H})\} = E\{X|\mathcal{G}\}$.

(12). Let $X_1, X_2, \ldots$ be i.i.d. nonnegative random variables with $E\{X_1\} = 1$. Let $R_n = \prod_{i=1}^{n} X_i$, and show that R_n is a martingale for the σ-algebras $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, where $n = 1, 2, \ldots$.