# AMS Foundation Exam - Part A, January 23, 2025

Name: ______________________________ ID Num. ____________________

LA: ______ / 30

AC: ______ / 30 Total: ______ / 60

This component of the Foundation Exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with three problems in each. Each question is worth 10 points; answer all **THREE** questions from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Show all work and/or justify your responses.

**Your solutions must be submitted within 5 minutes of the end of the exam. Late submissions – those received after 11:05 am EST – will not be scored.**

**Good Luck!**

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Signature**

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## Section 1: Linear Algebra

1. Consider the linear system of equations with real variables:

$$\begin{pmatrix} k & 1 & 2 \\ k & k & 1 \\ k^2 & k^2 & 2k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ q \end{pmatrix}.$$

   (a). For the coefficient matrix, analyze how the value of $k$ changes the dimension of the column space.

   (b). For what values of $k, p, q$, the equation has unique solution, no solution, and infinite number of solutions.

   (c). When the equation has infinite number of solutions, do these solutions form a subspace of $R^3$? Explain why or why not.

2. Consider the following subspaces of $R^4$:

$$U = span(u_1, u_2, u_3) = span\{(1, 3, -2, 2), (1, 4, -3, 4), (2, 3, -1, -2)\}$$
$$W = span(w_1, w_2, w_3) = span\{(1, 3, 0, 2), (1, 5, -6, 6), (2, 5, 3, 2)\}$$

(a). Determine if the following are subspaces of $R^4$?

$$U \cap W, \quad U \cup W, \quad U + W.$$

(b). If any of the above is a subspace of $R^4$, find its dimension and basis.

3. Given the matrix $A$ as

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(a). Find all eigenvalues and eigenvectors of $A$.

(b). Find $P$ and $P^{-1}$ such that $P^{-1}AP = \Lambda$, where $\Lambda$ is diagonal.

(c). Find $A^{10}$.

(d). Find $e^A$.

(e). For any given vector $x \in R^3$, analyze and find

$$\lim_{n\to\infty} \frac{||A^{n+1}x||}{||A^n x||}$$

## Section 2: Advanced Calculus

1. (a). Given $u(x,y) = \ln\left(\sqrt{x} + \sqrt{y}\right)$, calculate

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

(b). Calculate the limit

$$\lim_{n\to\infty} \frac{1}{n}\left(\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) \cdots + \ln\left(1+\frac{n}{n}\right)\right).$$

(c). Calculate the limit

$$\lim_{x\to 0}\left(\frac{\cos 2x - 1}{x^4 - x^2}\right)$$

2. If $0 < a < b$, use Cauchy mean value theorem to prove

$$1 - \frac{a}{b} \;<\; \ln\frac{b}{a} \;<\; \frac{b}{a} - 1$$

. Hint: consider functions $f(x) = \ln x$ and $g(x) = x$.

3. Consider the function

$$f(x, y, z) = x^2 + 4y^2 + 9z^2$$

using an appropriate coordinate transformation to evaluate the triple integral

$$\iiint_R e^{\sqrt{f(x,y,z)}} dxdydz$$

where $R$ is the volume defined by $f(x, y, z) \leq 16$.